NUMERICAL SOLUTION TO THE PROBLEM OF THE COMPLETE STABILIZATION OF A SUPERSONIC BOUNDARY LAYER

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A method is suggested for solving numerically the problem of the complete stabilization of a supersonic boundary layer. It is shown that when the surface is significantly cooled, the neutral stability curve splits into two branches. A calculation is given for the temperatures of complete stabilization for both neutral curves. A comparison of the results obtained with those derived from asymptotic calculations shows that above M = 2 (M is the Mach number) the asymptotic method gives incorrect results.

1. Lees [1] was the first to show that intensive cooling can ensure the complete stability of a supersonic laminar boundary layer to small two-dimensional perturbations. Calculations of the critical values of the surface temperatures required for the complete stabilization of a boundary layer at a flat plate have been made by an asymptotic method [2-6]. The authors of this paper are not aware of any numerical solution to the problem.

In the formulation of the asymptotic method used in all the above papers, it is required that the expansion parameter $\varepsilon = (\alpha R)^{-1/2}$, where α is the perturbation wave number and R is the Reynolds number, should be small in comparison with unity. However, for Mach numbers of 2-6, ε varies from 0.34 to 0.47, i.e., it is not small. Therefore the results obtained in these papers for M > 2 may be incorrect.

In this paper we suggest a method for solving numerically the problem of the complete stabilization of a boundary layer; this method does not require any restrictions on the value of αR.

2. From the asymptotic equations of Lees [1], which are valid for small supersonic Mach numbers, it can be shown that the values of αR on the upper and lower asymptotic curves of neutral stability tend to finite and different limits. As the surface temperature is reduced, the critical Reynolds number R* goes up and the two branches of the neutral curve get closer together, thus reducing the instability region. At a certain critical temperature T_w^* the branches merge and the values of αR on them coincide. Then there must exist in the (T_w, αR) plane a curve of values of αR on the asymptotes on which the minimum value of T_w gives the temperature of complete stabilization T_w^*. For lower temperatures in a laminar boundary layer of a viscous thermally-conducting gas, neutral or unstable supersonic perturbations cannot exist [1]. In order to construct the curve of the values of αR on the asymptotes, we use the Dun and Lin system of equations [4]. Since on the asymptotes we can take [1]

$$\alpha = 0,\ R = \infty,\ \alpha R = \text{const}$$

the Dun and Lin equations take the form

$$\begin{gathered}\rho^\circ [i(U^\circ - c)u + U_y{}^\circ v] + \frac{iP}{\gamma M^2} = \frac{\mu}{\alpha R} u_{yy} \\ P_y = 0 \\ i(U^\circ - c)\rho + \rho_y{}^\circ v + \rho^\circ (iu + v_y) = 0 \\ \rho^\circ [i(U^\circ - c)\theta + T_y{}^\circ v] + (\gamma - 1)(iu + v_y) = \frac{\gamma\mu}{\sigma\alpha R}\theta_{yy} \\ \frac{P}{P^\circ} = \frac{\theta}{T^\circ} - \frac{\rho}{\rho^\circ}\end{gathered} \tag{1}$$

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Here U°, ρ°, and T° are the time-averaged velocity, density and temperature; u, αv, ρ, θ, and P are the perturbations in the longitudinal and transverse velocities, density, temperature, and pressure; y is the distance along the normal to the surface, the subscript y indicates differentiation, γ is the adiabatic constant, M is the Mach number, σ is the Prandtl number, μ is the viscosity, R is the Reynolds number, α is the wave number of the perturbation, and $c = c_r + ic_i$ is the perturbation phase velocity (at the asymptotes $c = 1 - M^{-1}$). It is assumed that the perturbations vary with the longitudinal coordinate x and the time t in the form $\exp i\alpha(x - ct)$. A similar limiting transition for calculating the values of αR on the asymptotes of the neutral stability surface for incompressible liquid flow is used in [7].

The boundary conditions for the system (1) can be taken in the form

$$u(0) = v(0) < \theta_y(0) = 0; \tag{2}$$

u, v, and θ are bounded as $y \to \infty$. It would appear that the more correct boundary condition on temperature perturbations should be $\theta(0) = 0$. The choice of the condition $\theta_y(0) = 0$ can be explained by the fact that this case has been better studied by asymptotic methods. We can therefore make a detailed comparison of the results and give a more correct decision on the applicability of the asymptotic methods.

By introducing the variables

$$z_1 = u, \; z_2 = u_y, \quad z_3 = v, \; z_4 = P/\gamma M^2, \; z_5 = \theta, \; z_6 = \theta_y$$

we can reduce the system (1) to the six first-order equations

$$z_{iy} = \sum_{j=1}^{6} G_{ij} z_j, \qquad (i = 1, \ldots, 6) \tag{3}$$

with the boundary conditions

$$z_1(0) = z_3(0) = z_6(0) = 0 \tag{4}$$

and z_1, z_3, and z_5 are bounded as $y \to \infty$.

Outside the boundary layer the system coefficients are constants, and the solution has the form

$$z_i = \sum_{j=1}^{6} C_j A_i^{(j)} e^{\lambda_j y}$$

where λ takes the values

$$\lambda_{1,2} = \pm\sqrt{i\alpha R/M}, \qquad \lambda_{3,4} = \pm\sqrt{i\alpha R\sigma/M}, \qquad \lambda_{5,6} = 0.$$

Since the solutions with λ_1 and λ_3 do not satisfy the boundary conditions at infinity, C_1 and C_3 must be identically equal to zero. The characteristic vectors $A^{(2)}$ and $A^{(4)}$ have only three nonzero components each:

$$\begin{aligned} &A_1^{(2)} = 1, \quad A_2^{(2)} = \lambda_2, \quad A_3^{(2)} = -i/\lambda_2 \\ &A_3^{(4)} = i/M, \quad A_5^{(4)} = \lambda_4, \quad A_6^{(4)} = \lambda_4^2. \end{aligned} \tag{5}$$

The characteristic vectors corresponding to λ_5 and λ_6 are conveniently taken as

$$\begin{aligned} &A_1^{(6)} = -M, \quad A_2^{(6)} = A_6^{(6)} = 0 \\ &A_3^{(6)} = C_5, \quad A_4^{(6)} = 1, \quad A_5^{(6)} = (\gamma - 1)M^2. \end{aligned} \tag{6}$$

Comparing the solution of (3) on the edge of the boundary layer (5) and (6) with the solution of the complete system of stability equations for $y = \delta$ (δ is the thickness of the boundary layer) with $\alpha = 0$, $c = 1 - M^{-1}$ [8], we can see that

$$C_5 = \lim_{\alpha \to 0,\, c \to 1 - M^{-1}} \frac{\sqrt{1 - M^2(1 - c)^2}}{\alpha}.$$

The existence of such a limit has been shown in the asymptotic solution of the problem of complete stabilization [9].

Each of the three vectors $A^{(j)}$ is used as initial data in the numerical integration of (3) from the external surface of the boundary layer to the wall. If the parameter in front of the leading derivative $(\alpha R)^{-1}$ becomes small, the numerical integration can be carried out by the orthogonalization method of Godunov [10]. During the calculation the values of αR, T_w, and C_5 were chosen so as to satisfy for fixed M the homogeneous boundary conditions on the surface of the plate [11].

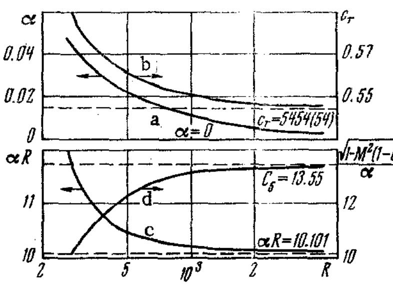

Fig. 1

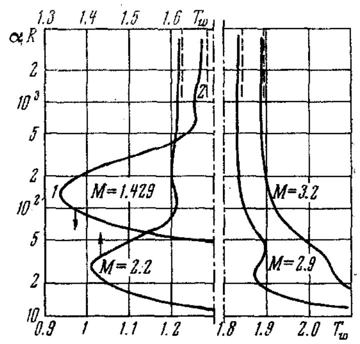

Fig. 2

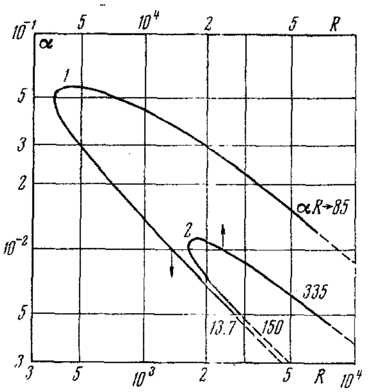

Fig. 3

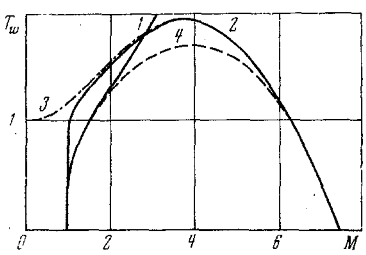

Fig. 4

3. We took the values $\sigma = 0.75$, $\gamma = 1.4$, and $\mu = T^{\circ}$. In order to determine the U°, T°, and ρ° – the velocity, temperature, and density distribution of the main flow – the equations of a laminar boundary layer on a flat plate were also integrated numerically.

Figure 1a shows the lower branch of the neutral stability curve for M = 2.2 and $T_W = 1.82$; Fig. 1b, c, and d show the behavior of c_r, αR, and $\sqrt{1 - M^2(1-c)^2}/\alpha$ with increase in R. The limiting values of αR and $\sqrt{1 - M^2(1-c)^2}/\alpha$ were obtained by numerical integration of (1).

Curves showing the variation of αR on the asymptotes with surface temperature are given in Fig. 2 for various Mach numbers. For M = 1.429 and 2.2, there is a temperature region in which four values of αR correspond to each value of T_W; this means that two neutral stability curves can exist. Both of these curves were calculated from the Dan and Lin system with M = 2.2 and $T_W = 1.606$ (Fig. 3). At the surface temperature for which there are three corresponding values of αR, these curves merge into each other. The second curve 2 rapidly disappears as the temperature is reduced. The first neutral curve 1 continues to exist over a certain temperature range. As αR increases, the curves in Fig. 1 have asymptotes at a certain temperature. It can be shown from the Lees and Lin [9] equations that this is the temperature of complete nonviscous stabilization.

As the Mach number M increases, the complete stabilization temperature of the second neutral curve 2 approaches the complete nonviscous stabilization temperature, and the corresponding temperature of the first curve 1 reaches a value even higher than this. When M = 3.2, the first neutral curve disappears altogether, and only one curve remains, for which at the asymptotes

$$\alpha \to 0, \quad c \to 1 - M^{-1}, \quad \alpha R \to \text{const}.$$

The division into first and second neutral curves is quite arbitrary but it does enable us to follow the changes in the instability region as T_W and M vary.

Figure 4 shows the variation in the complete stability temperature with M. The numbers 1 and 2 indicate the temperatures for the first and second neutral curves that we have obtained here. Curve 3 is the temperature of complete nonviscous stabilization [3] and curve 4 shows the results obtained by the asymptotic method [4].

The complete stabilization temperatures of the first curve agree with the asymptotic calculations up to M = 1.8. Above M = 2, there is not even qualitative agreement. The error which appears in the asymptotic methods is not only due to the decrease in αR (increase in $\varepsilon = (\alpha R)^{-1/2}$). Despite the fact that above M = 2.7 the parameter αR obtained by numerical calculations begins to increase, the difference between the numerical and asymptotic results continues to get bigger. As M increases, the temperature perturbations produce a greater effect on the variations in velocity, and this fact is not taken into account in the asymptotic methods.

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